Name: | | 50 |

General Exam Summer 2007

Algebra

August 20, 2007

Rules

1. This is a closed book exam. To use a result, you can cite it by name (e.g., say "by the Main Theorem on Finitely Generated Modules over PIDs") or, if the result does not have a common name, you can just restate it (e.g., say "We know from class that the polynomial ring over a UFD is a UFD").
2. Make sure that I can understand what you write. It will help if you use complete sentences to communicate your ideas. I do not engage in reading minds. You really have to tell me what is going on. Make sure that it is impossible to misunderstand your write-up. I can be somewhat dense, and that will work to your disadvantage.
3. If you think a problem needs clarification, please ask. I will respond to the class.
4. There is a total of 50 points on this exam.
5. This exam has 9 problems and 10 pages. Make sure that none are missing in your copy.

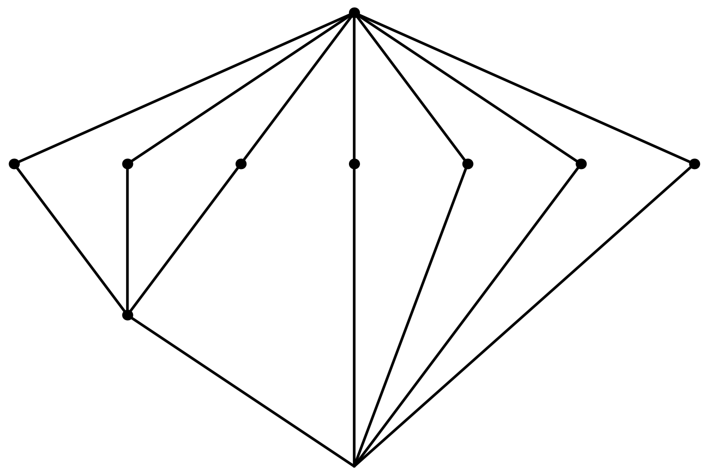
Hints

1. Problems are not sorted according to difficulty.

Room for your pledge:

Problem 1 [5 points]
Prove or disprove: the alternating group $\mathbf{A}_{2007}$ has a generating set that consists entirely of elements of order 2007.

Problem 2 [5 points]
Let M/K be a Galois extension of degree 12. The following is the lattice of intermediate extensions.



Here, vertices of the same hight may correspond to intermediate extension of different degree. The top vertex corresponds to M and the bottom vertex corresponds to K.

Show that the Galois group of M/K is $\mathbf{A}_4$.

Problem 3 [5 points]
The aim of this problem is to show that finite non-abelian simple groups do not arise as subgroups of $\mathrm{GL}_2(\mathbb{C})$. You may use the Feit-Thompson theorem that says that finite non-abelian simple groups have even order.

1. Show that there is exactly one element of order 2 in $\mathrm{SL}_2(\mathbb{C})$ and determine that matrix.
2. Show that any finite non-abelian simple subgroup of $\mathrm{GL}_2(\mathbb{C})$ is contained in $\mathrm{SL}_2(\mathbb{C})$.
3. Deduce that $\mathrm{GL}_2(\mathbb{C})$ does not contain any finite non-abelian simple subgroups.

Problem 4 [5 points]
Let $M := \mathbb{C}(t)$ be the rational function field, and consider the two $\mathbb{C}$-automorphisms φ and ψ of M defined by $\varphi(t) = it$ and $\psi(t) = t^{-1}$.

1. Determine the group $G := \langle \varphi, \psi \rangle \leq \mathrm{Aut}_{\mathbb{C}}(M)$ generated by φ and ψ.
2. Let $K := \mathrm{Fix}_M(G)$ be the fixed field of G. Show that $G = \mathrm{Aut}_K(M)$ and that $t^4 + t^{-4}$ generates K as a field extension over $\mathbb{C}$.

Problem 5 [5 points]
Let A be an $n \times n$ square matrix with rational coefficients. Suppose that A has order 5 and assume that the equation $A\mathbf{v} = \mathbf{v}$ is only satisfied for $\mathbf{v} = 0$. Show that n is divisible by 4.

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Problem 6 [5 points]
Show that there is some $n \in \mathbb{N}$ so that $\mathbb{Z}[in]$ is not a PID.

Problem 7 [5 points]
Show that

$$p(x) = x^3 - 3x^2 + 15x - 7$$

is irreducible in $\mathbb{Q}[x]$.

Problem 8 [5 points]
Let M be the $\mathbb{Z}$-submodule of $\mathbb{Z}^2$ generated by

$$\begin{pmatrix} 2 \\ 10 \end{pmatrix}, \quad \begin{pmatrix} -2 \\ 8 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \quad \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

Find a basis for M and determine the structure of $\mathbb{Z}^2/M$ as an Abelian group.

Problem 9 [10 points]
For the following, no reasoning is required:

1. True or false: in a PID, irreducible elements are prime.
2. True or false: in a domain, irreducible elements are prime.
3. True or false: in a PID, prime elements are irreducible.
4. True or false: in a domain, prime elements are irreducible.
5. True or false: a solvable-by-solvable group is solvable.
6. Compute $(\mathbb{Z}_2 \oplus \mathbb{Q}) \otimes_{\mathbb{Z}} (\mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z})$.
7. State the definition of a prime ideal.
8. State Hilbert's basis theorem for polynomial rings.
9. State the definition of a solvable group.
10. State the definition of a Euclidean ring.