General Exam Summer 2006

Algebra

August 14, 2006

Rules

1. This is a closed book exam. To use a result, you can cite it by name (e.g., say “by the Main Theorem on Finitely Generated Modules over PIDs”) or, if the result does not have a common name, you can just restate it (e.g., say “We know from class that the polynomial ring over a UFD is a UFD”).

2. Make sure that I can understand what you write. It will help if you use complete sentences to communicate your ideas. I do not engage in reading minds. You really have to tell me what is going on. Make sure that it is impossible to misunderstand your write-up. I can be somewhat dense, and that will work to your disadvantage.

3. If you think a problem needs clarification, please ask. I will respond to the class.

4. There is a total of 0 points on this exam.

5. This exam has 0 problems and 0 pages. Make sure that none are missing in your copy.

Hints

1. Problems are not sorted according to difficulty.

Room for your pledge:
Problem 1  [5 points]
Prove that there is no simple group of order \(2 \times 3^3 \times 5^2\).
Problem 2  [5 points]
Let $G$ be a finite group wherein any two conjugate elements commute. Prove that $G$ is solvable. (More is true: $G$ actually has to be nilpotent; that is, however, a little harder to show.) There is partial credit for proving that $G$ is not simple unless it is Abelian.
Problem 3  [5 points]
Show that any nilpotent group $G$ of order 900 is Abelian.
Problem 4  [5 points]

Decide whether

\[ xy^2 + x^2 y + 2xy + y + x + 1 \]

is irreducible in \( \mathbb{Q}[x, y] \).
Problem 5  [5 points]
Let $R$ be a commutative ring. A radical is an ideal $I \subseteq R$ such that, for any $a \in R$, we have $a \in I$ whenever some power $a^k \in I$.

1. (2 points) Show that every prime ideal is a radical.

2. (3 points) Assume $I$ is a radical and $a \in R$ does not lie in $I$. Show that there exists a prime ideal $P$ that contains $I$ but does not contain $a$. (Hint: use Zorn’s lemma)
Problem 6  [5 points]
Over \( \mathbb{C} \), find the RCF, JCF, the elementary divisors, the invariant factors, the characteristic and the minimal polynomial of:

\[
\begin{pmatrix}
0 & -1 & 3 \\
1 & 2 & -3 \\
1 & 1 & -2
\end{pmatrix}
\]
Problem 7  [5 points]
Determine the number of monic irreducible polynomials of degree 2 in $\mathbb{F}_7[x]$. 
Problem 8  [5 points]
Let $M/K$ be a Galois extension of degree 270. Show that there is an intermediate extension $M/L/K$ with $[L : K] = 30$. 

Problem 9  [5 points]
Let $M/K$ be field extension and let $\zeta \in M - K$ be algebraic over $K$. Let $L := K(\zeta)$ denote the intermediate field generated by $\zeta$. Show that $L \otimes_{K[x]} K[[x]] = \{0\}$. Here, we consider $L$ as a $K[x]$-module where we let $x$ as multiplication by $\zeta$. 

General Exam in Algebra
Problem 10  [5 points]
For the following questions, no reasoning is required:

- True or false: if a group is solvable, then is is nilpotent.
- State the universal property for the tensor product $A_1 \otimes \mathbb{Z} A_2$ of two Abelian groups.
- True or false: every finite simple group has odd order.
- Give the definition of when a short exact sequence of groups
  \[ 1 \rightarrow N \xrightarrow{\iota} G \xrightarrow{\pi} Q \rightarrow 1 \]
splits.
- True or false: Every Euclidean ring is a UFD.