Name: | | 0 |

General Exam Summer 2006

Algebra

August 14, 2006

Rules

1. This is a closed book exam. To use a result, you can cite it by name (e.g., say "by the Main Theorem on Finitely Generated Modules over PIDs") or, if the result does not have a common name, you can just restate it (e.g., say "We know from class that the polynomial ring over a UFD is a UFD").
2. Make sure that I can understand what you write. It will help if you use complete sentences to communicate your ideas. I do not engage in reading minds. You really have to tell me what is going on. Make sure that it is impossible to misunderstand your write-up. I can be somewhat dense, and that will work to your disadvantage.
3. If you think a problem needs clarification, please ask. I will respond to the class.
4. There is a total of 0 points on this exam.
5. This exam has 0 problems and 0 pages. Make sure that none are missing in your copy.

Hints

1. Problems are not sorted according to difficulty.

Room for your pledge:

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Problem 1 [5 points]
Prove that there is no simple group of order $2 \times 3^3 \times 5^2$.

Problem 2 [5 points]
Let G be a finite group wherein any two conjugate elements commute. Prove that G is solvable. (More is true: G actually has to be nilpotent; that is, however, a little harder to show.) There is partial credit for proving that G is not simple unless it is Abelian.

Problem 3 [5 points]
Show that any nilpotent group G of order 900 is Abelian.

Problem 4 [5 points]
Decide whether

$$xy^2 + x^2y + 2xy + y + x + 1$$

is irreducible in $\mathbb{Q}[x, y]$.

Problem 5 [5 points]
Let R be a commutative ring. A radical is an ideal $I \trianglelefteq R$ such that, for any $a \in R$, we have $a \in I$ whenever some power $a^k \in I$.

1. (2 points) Show that every prime ideal is a radical.

2. (3 points) Assume I is a radical and $a \in R$ does *not* lie in I. Show that there exists a prime ideal P that contains I but does not contain a. (Hint: use Zorn's lemma)

Problem 6 [5 points]
Over $\mathbb{C}$, find the RCF, JCF, the elementary divisors, the invariant factors, the characteristic and the minimal polynomial of:

$$\begin{pmatrix} 0 & -1 & 3 \\ 1 & 2 & -3 \\ 1 & 1 & -2 \end{pmatrix}$$

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Problem 7 [5 points]
Determine the number of monic irreducible polynomials of degree 2 in $\mathbb{F}_7[x]$.

Problem 8 [5 points]
Let M/K be a Galois extension of degree 270. Show that there is an intermediate extension $M/L/K$ with $[L : K] = 30$.

Problem 9 [5 points]
Let M/K be field extension and let $\zeta \in M - K$ be algebraic over K. Let $L := K(\zeta)$ denote the intermediate field generated by ζ. Show that $L \otimes_{K[x]} K[[x]] = \{0\}$. Here, we consider L as a $K[x]$-module where we let act x as multiplication by ζ.

Problem 10 [5 points]
For the following questions, no reasoning is required:

- True or false: if a group is solvable, then is is nilpotent.
- State the universal property for the tensor product $A_1 \otimes_{\mathbb{Z}} A_2$ of two Abelian groups.
- True or false: every finite simple group has odd order.
- Give the definition of when a short exact sequence of groups

  $$1 \to N \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1$$

  splits.
- True or false: Every Euclidean ring is a UFD.