Algebra General Exam 2004

August 16, 2004, UVA

(1) (10 points) Assume that the group G is a direct product of two finite subgroups A and B, $G = A \times B$, and that $|A|$, $|B|$ are relatively prime. Show that $H = (H \cap A) \times (H \cap B)$ for any subgroup H of G.

(2) (14 points) Let G be a group of order 56, and let P_p for $p \in \{2, 7\}$ be a Sylow p-subgroup of G.
(a) (6 points) Show that P_2 or P_7 is normal in G.
(b) (3 points) Give an example of a group G with $|G| = 56$ where P_2 is not normal.
(c) (5 points) Show that there exists a group G of order 56 with a non-normal P_7. You can either do this by exhibiting a concrete example with this property or by describing how to construct such a group. In the latter case you have to justify why your approach works but you needn't give all details of the construction.

(3) (12 points, 6 points each) Consider the ring $R = \mathbb{Z}[\sqrt{-7}] = \{m + n\sqrt{-7} \mid m, n \in \mathbb{Z}\}$.
(a) Is R a UFD? Give arguments for your answer.
(b) Exhibit an ideal I in R which is not principal. *Show* that your I is not principal.

(4) (12) Let L/K be a finite Galois extension. Suppose there exists an element $\alpha \in L$ and another root α' of the minimal polynomial $\mu_{\alpha|K}$ of α over K such that the difference $\alpha' - \alpha$ is an element of $K \setminus \{0\}$.
(a) (9 points) Prove that the characteristic p of K is different from 0 and that p divides $[L : K]$.
(b) (3 points) Give an example of an extension L/K and elements α, α' as described above.

(5) (10 points, 5 points each) Let N be the $\mathbb{Z}$-submodule of $\mathbb{Z}^3$ generated by the column vectors $(2, 2, -2)^t, (-4, -2, 4)^t$ and $(2, 4, 4)^t \in \mathbb{Z}^3$.
(a) Determine the structure of the abelian group $\mathbb{Z}^3/N$.
(b) Determine a basis y_1, y_2, y_3 of $\mathbb{Z}^3$ and natural numbers $d_1 \mid d_2 \mid d_3$ such that d_1y_1, d_2y_2, d_3y_3 is a $\mathbb{Z}$-basis of N.

(6) (10 points) Let K be an arbitrary field ($\rightarrow$ case distinction!). Classify, up to similarity, all matrices $A \in GL_4(K)$ of order 2. (Use an appropriate canonical form.)

(7) (12 points) Consider the group $G = SL_2(\mathbb{F}_4)$.
(a) (4 points) Show *without* specifying any matrix that G contains an element of order 5.
(b) (8 points) Exhibit a concrete matrix $A \in SL_2(\mathbb{F}_4)$ of order 5. Use $\mathbb{F}_4 = \{0, 1, \alpha, \alpha^2\}$, where α is a root of $x^2 + x + 1$. Describe in detail how you obtained A; you shouldn't just guess!
(Hint: You might first factorize $x^5 - 1$ in $\mathbb{F}_4[x]$.)

(8) (8 points) Let K be a field, V a finite-dimensional vector space over K, and v, w two non-zero vectors in V. Show that $v \otimes w = w \otimes v$ in $V \otimes_K V$ if and *only if* there exists a $c \in K^*$ such that $w = cv$.

(9) (12 points, 3 points each) Decide in each of the following four cases whether the given statement is true or false. You need not give any arguments.

(a) If $n \geq 3$ is a natural number and $A = J_n(0) \in M_n(\mathbb{R})$ is a Jordan block matrix of size $n \times n$ with eigenvalue 0, then $J_2(0) \oplus J_{n-2}(0)$ (i.e. two Jordan blocks, one of size 2 and one of size $n - 2$) is the Jordan canonical form of A^2.

(b) If R is an integral domain which is also a finite-dimensional K-algebra for some field $K \subseteq R$, then R is itself a *field*.

(c) Let L/K be a field extension, and assume that L contains a primitive n^{th} root of unity ζ_n. Then M/K is *normal* for any subfield M of $K(\zeta_n)$ which contains K (i.e. we are considering the tower $L/K(\zeta_n)/M/K$).

(d) Let V be a finite-dimensional vector space over a field K and $B : V \times V \to K$ a symmetric K-bilinear form. If there exists a subspace $W \neq \{0\}$ of V such that $V = W \oplus W^\perp$, where $W^\perp := \{x \in V \mid B(x, w) = 0 \text{ for all } w \in W\}$, then B is *nondegenerate*.