Syllabus for Algebra General Exam

I Groups

Basic notions. Subgroups, homomorphisms, normal subgroups, quotients, etc. Coset decomposition and the Lagrange theorem. First and second isomorphism theorems. Normalizers, centralizers, centers, commutators, automorphism groups. Action of a group on a set; examples: action by conjugation and permutation action on cosets. Class equation. Groups acting on groups; application: semidirect products. Classes of groups: simple groups, solvable, and nilpotent groups. Defining groups by generators and relations.

Further topics. Finite p-groups and their properties. Sylow theorems: proof and applications.

II Rings and Modules

Basic notions. Ideals, quotient rings, homomorphisms, prime ideals, and maximal ideals. Euclidean domains, principal ideal domains, and unique factorization, submodules, quotient modules, free modules.

Further topics. Localization and the field of fractions of a domain; local rings. Unique factorization in polynomial rings of several variables. Structure of finitely generated modules over a principal ideal domain; applications: free Abelian groups and the existence of the Jordan canonical form. Tensor products over commutative rings. Multilinear algebra.

III Advanced Linear Algebra

Basic notions. Vector spaces, bases, dimension, subspaces, linear maps, matrices, characteristic polynomial, eigenvalues/eigenvectors, diagonalization.

Further topics. The Jordan canonical form. Algorithm for finding the Jordan canonical form of a matrix. The Cayley-Hamilton Theorem. Minimal polynomial. Rational canonical form. Bilinear and quadratic forms. Orthogonal bases and diagonalization. The law of inertia and classification of quadratic forms over R: Diagonalization of real quadratic forms by orthogonal transformations. Orthogonal groups. Hermitian forms and unitary groups.

V Fields

Finite and algebraic extensions of fields. Algebraic closure. The splitting field of a polynomial; normal extensions. Separable extensions. The theorem on a primitive element. Finite fields. Fundamental theorem of Galois theory.