Complexity, Combinatorial Positivity, and Newton Polytopes

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**Poorly understood issue:** Why are some decision problems have fast algorithms and others seem to need costly search?

Some complexity classes:
- **NP:** LP ($\exists x \geq 0, Ax=b$?)
- **coNP:** Primes
- **P:** LP and Primes!
- **NP-complete:** Graph coloring

Famous theoretical computer science problems:
- $P \overset{?}{=} NP$
- $NP \overset{?}{=} coNP$
- $NP \cap coNP \overset{?}{=} P$
In algebraic combinatorics and combinatorial representation theory we often study:

\[
F_\diamond = \sum_{\alpha} c_{\alpha,\diamond} x^\alpha = \sum_{s \in S} \text{wt}(s) \in \mathbb{Z}[x_1, \ldots, x_n]
\]

**Example 1:** \(\diamond = \lambda \implies F_\diamond = s_\lambda \) (Schur), \(c_{\alpha, G} = \text{Kostka coeff.}\).

**Example 2:** \(\diamond = G = (V, E) \implies F_\diamond = \chi_G \) (Stanley’s chromatic symmetric polynomial), \(c_{\alpha, G} = \#\text{proper colorings of } G \text{ with } \alpha_i\text{-many colors } i\).

**Example 3:** \(\diamond = w \in S_\infty \implies F_\diamond = \mathcal{G}_w \) (Schubert polynomial).

More later.
**Nonvanishing**: What is the complexity of deciding $c_{\alpha, \diamond} \neq 0$ as measured in the length of the input $(\alpha, \diamond)$ assuming arithmetic takes constant time?

- In general **undecidable**: Gödel incompleteness ’31, Turing’s halting problem ’36.
- Our cases of interest have combinatorial positivity:
  \[ \exists \text{ rule for } c_{\alpha, \diamond} \in \mathbb{Z}_{\geq 0} \implies \text{Nonvanishing}(F_{\diamond}) \in \text{NP}. \]
Evidently, nonvanishing concerns the *Newton polytope*,

$$\text{Newton}(F_\diamond) = \text{conv}\{\alpha : c_{\alpha,\diamond} \neq 0\} \subseteq \mathbb{R}^n.$$ 

- Monical-Tokcan-Y. '17: $F_\diamond$ has *saturated Newton polytope* (SNP) if $\beta \in \text{Newton}(F_\diamond) \iff c_{\beta,\diamond} \neq 0$
- Many polynomials have this property.

**Importance of SNP property:**

**Observation 1:** SNP $\Rightarrow$ nonvanishing($F_\diamond$) is equivalent to checking membership of a lattice point in Newton($F_\diamond$).

**Observation 1’:** SNP $+ “efficient”$ halfspace description of Newton($F_\diamond$) $\implies$ nonvanishing($F_\diamond$) $\in$ coNP.

$\therefore$ in many cases nonvanishing($F_\diamond$) $\in$ NP $\cap$ coNP.
Nonvanishing and NP

Example 1': $s_\lambda$ has SNP. Newton($s_\lambda$) = $P_\lambda$ (the permutahedron). Nonvanishing($s_\lambda$) ∈ P by dominance order (Rado’s theorem).

Example 2': $\chi_G$ does not have SNP.

coloring ∈ NP-complete $\implies$ Nonvanishing($\chi_G$) ∈ NP-complete.

$\therefore$ nonvanishing hits the extremes of NP.

Question: What about the nonextremes?

- Many problems suspected of being NP-intermediate: e.g.,
  graph isomorphism, factorization
- Ladner’s theorem: P $\neq$ NP $\implies$ NP − intermediate $\neq$ ∅
- NP $\cap$ coNP is important to this discussion:

  coNP $\cap$ NP − complete $\neq$ ∅ $\implies$ NP = coNP!

- This is why factorization is not expected to be NP-complete.
- Most public key cryptography relies on NP $\cap$ coNP $\neq$ P.
**Conjecture 1:** [Stanley '95] If $G$ is claw-free (i.e., it contains no induced $K_{1,3}$ subgraph), then $\chi_G$ is Schur positive.

**Conjecture 2:** [C. Monical '18] If $\chi_G$ is Schur positive, then it is SNP.

**Conjecture 1+2:** If $G$ is claw-free then $\chi_G$ is SNP.

**Theorem:** (Holyer '81) Coloring of claw-free $G$ is NP-complete.

**Corollary:** nonvanishing$(\chi_{\text{claw-free}} G)$ $\in$ NP-complete.

$\therefore$ Conjecture 1+2 and a halfspace description of

$\text{Newton}(\chi_{\text{clawfree}} G) \implies \text{NP} = \text{coNP}$

Suggests a new complexity-theoretic rationale for the study of $\chi_G$. 
In many cases of algebraic combinatorics, \( \{F_{\diamond}\} \) has combinatorial positivity and SNP. If one also has an efficient halfspace description of Newton(\( F_{\diamond} \)), then nonvanishing(\( F_{\diamond} \)) \( \in \) NP \( \cap \) coNP.

Four possible outcomes of such a study:

(I) **Unknown**: it is an open problem to find additional problems that are in NP \( \cap \) coNP that are not known to be in P.

(II) **P**: Give an algorithm. It will likely illuminate some special structure, of independent combinatorial interest.

(III) **NP-complete**: proof solves \( \text{NP} \equiv \text{coNP} \) with “\( = \)”.

(IV) **NP-intermediate**: proof solves \( \text{NP-intermediate} \equiv \emptyset \) with “\( \neq \)”, i.e., P \( \neq \) NP.

Next: do this for Schubert polynomials (outcomes (I) and (II)).
$B$ acts on $GL_n/B$ with \textit{finitely many orbits}, the Schubert cells, whose closures $X_w$, $w \in S_n$ are the \textbf{Schubert varieties}.

Lascoux and Schützenberger’s (1982) main idea in type $A$ (after Bernstein-Gelfand-Gelfand):

- Pick $\mathcal{S}_{w_0} = x_1^{n-1}x_2^{n-2} \cdots x_{n-1}$ as an especially nice representative of the class of a point
- Apply \textit{Newton’s divided difference operator}

\[ \partial_i f = \frac{f - f^{s_i}}{x_i - x_{i+1}}, \]

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\[
\text{to recursively define all other } \mathcal{S}_w \text{ using weak Bruhat order.}
\]

This starts the theory of \textit{Schubert polynomials}. 


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There are many combinatorial rules that establish that $c_{\alpha, w} \in \mathbb{Z}_{\geq 0}$. However, none of these prove nonvanishing $S_w \in P$ since they involve exponential search.

**Theorem A:** (Adve-Robichaux-Y. '18) $c_{\alpha, w}$ is $\#P$-complete.

∴ no polynomial time algorithm to compute $c_{\alpha, w}$ exists unless $P = NP$.

Counting is hard, nonvanishing is easy:

**Theorem B:** (Adve-Robichaux-Y. '18) nonvanishing $S_w \in P$

**Analogy:** Computing the permanent of a 0, 1-matrix is $\#P$-complete but nonzeroness is easy (Edmonds-Karp matching algorithm).
A tableau rule for nonvanishing

Fillings of the Rothe diagram of 31524:

Theorem C: (Adve-Robichaux-Y. ’18)

\[ c_{\alpha,w} \neq 0 \iff \text{Tab}(w, \alpha) \neq \emptyset. \]
The Schubitope $S_D$ was introduced by Monical-Tokcan-Y. ’17 for any $D \subseteq [n]^2$.

We give a generalization of tableau of Theorem C to any $D$.

Then introduce a new polytope $T_D$ whose integer points biject with tableaux.

Integer linear programming is hard but $T_D$ is totally unimodular. Now use LPfeasibility $\in P$.

Link to Schubert polynomials: For $D = D(w)$, Monical-Tokcan-Y. ’17 conjectured $S_D = \text{Newton}(\mathcal{G}_w)$. Proved by Fink-Mészáros-St. Dizier ’18.

First proved that nonvanishing($\mathcal{G}_w$) $\in \text{NP} \cap \text{coNP}$ hinting $\in P$.

NP and $\#P$ proof via transition.
In this talk we described an algebraic combinatorics paradigm for complexity on theoretical computer science.

Conversely, complexity gives some new perspectives on algebraic combinatorics.

In our main example, we obtain new results about Schubert polynomials and the Schubitope.