Stability of the centers of group algebras of $GL_n(q)$

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Outline

1. Stability of Symmetric Groups
2. Stability for $GL_n(q)$
3. Conjectures and Questions
Stability for symmetric groups

**Modified type**

- Conjugacy classes of symmetric group $S_n \Leftrightarrow \text{Par}_n = \{\text{partitions of } n\}$

  $n = 6$. $\sigma = (1, 3)(2, 4, 5, 6) \leadsto \text{type}$

  $n = 7$. $\sigma$ again, $\leadsto \text{type}$

- Problem: same $\sigma$ in $S_n$ and $S_{n+1}$, different cycle type.

- Solution: delete the first (=green) column.

- Call the remaining partition, $\text{type}$, the modified type of $\sigma$
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Class sums

- $\sigma$ has modified type $\lambda \Rightarrow |\lambda| = \text{length} \ell(\sigma) := \text{minimal length for } \sigma \text{ as a product of transpositions.}$

- $C_\lambda(n)$: conjugacy class of $S_n$ of modified type $\lambda$ (if $|\lambda| + \ell(\lambda) \leq n$)

- $c_\lambda(n)$: class sum of the class $C_\lambda(n)$ (if $|\lambda| + \ell(\lambda) \leq n$); otherwise $= 0$.

- Center of the group algebra, $\mathcal{Z}(\mathbb{Z}S_n)$, has a $\mathbb{Z}$-basis $\{c_\lambda(n) \mid \lambda \in \text{Par}\} \setminus \{0\}$.

(Here $\text{Par} = \bigcup_n \text{Par}_n$.)
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Stability for symmetric groups

An example of structure constants

Write the multiplication in the center $\mathcal{Z}(\mathbb{Z}S_n)$ as

$$c_\lambda(n)c_\mu(n) = \sum_\nu g_{\lambda\mu}^\nu(n)c_\nu(n), \quad \text{for } g_{\lambda\mu}^\nu(n) \in \mathbb{N}.$$ 

Example

$c_{(1)}(n) := \text{class sum of transpositions (=reflections) } (i, j) \text{ in } S_n.$

$$c_{(1)}(n) c_{(1)}(n) = n(n - 1)/2 c_{\emptyset}(n) + ?? c_{(1,1)}(n) + ??? c_{(2)}(n)$$

(independent of n)
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Stable structure constants

**Theorem (Farahat-Higman’59)**

1. $g_{\lambda \mu}^\nu(n)$ is polynomial in $n$
2. $g_{\lambda \mu}^\nu(n) = 0$ unless $|\nu| \leq |\lambda| + |\mu|$
3. If $|\nu| = |\lambda| + |\mu|$, then $g_{\lambda \mu}^\nu(n) = g_{\lambda \mu}^\nu$ is independent of $n$

- **Application**: modular representation theory of $S_n$
- **Connections**: Jucys-Murphy elements
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Conjectures and questions

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A stable ring

- $S_n$ and the center $\mathbb{Z}(\mathbb{Z}S_n)$ admits a filtration by $\ell(\sigma)$ (minimal length for $\sigma$ as a product of transpositions), $\forall \sigma \in S_n$.

Theorem (Farahat-Higman reformulated)

1. The associated graded $\mathbb{Z}^{gr}(\mathbb{Z}S_n)$ has structure constants independent of $n$: $c_\lambda(n)c_\mu(n) = \sum_{|\nu| = |\lambda| + |\mu|} g_{\lambda\mu}^{\nu} c_\nu(n)$

2. $\exists$ a stable center $``\mathbb{Z}^{gr}(\mathbb{Z}S_\infty)``$ with basis $\{c_\lambda | \lambda \in \text{Par}\}$ and $c_\lambda c_\mu = \sum_{|\nu| = |\lambda| + |\mu|} g_{\lambda\mu}^{\nu} c_\nu$

3. $\exists$ an epi $``\mathbb{Z}^{gr}(\mathbb{Z}S_\infty)`` \longrightarrow \mathbb{Z}^{gr}(\mathbb{Z}S_n)$, $c_\lambda \mapsto c_\lambda(n)$.

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1. The stable center $\mathbb{Q} \otimes_{\mathbb{Z}} ``\mathbb{Z}^{gr}(\mathbb{Z}S_\infty)``$ is a polynomial algebra in $c_{(r)}$, $r \geq 1$ and $``\mathbb{Z}^{gr}(\mathbb{Z}S_\infty)`` \cong \Lambda$.

2. $\mathbb{Z}^{gr,*}(\mathbb{Z}S_n) \cong H^{2*}(\text{Hilb}^n(\mathbb{C}^2), \mathbb{Z})$, cohomology ring of Hilbert scheme of $n$ points on $\mathbb{C}^2$ [Lehn-Sorger, Vasserot].
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Wreath products

• $\Gamma$: a finite group

• $\Gamma_n := \Gamma^n \rtimes S_n$ – a wreath product

• [Wang’04]. Generalization of Farahat-Higman stability to $\Gamma_n$.

• Let $\Gamma \leq SL_2(\mathbb{C})$. $\mathbb{Z}^{gr,*}(\mathbb{Z}\Gamma_n) \cong H^{2*}(\text{Hilb}^n(\mathbb{C}^2//\Gamma))$, cohomology ring of Hilbert scheme of $n$ points on the surfaces $\mathbb{C}^2//\Gamma$

• Analogous stability for
  (i) cohomology ring of Hilbert scheme of $n$ points of more general surfaces [Li-Qin-Wang’04]
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Reflection filtration on $GL_n(q)$

- $G_n := GL_n(q) = \{ g \in \text{Mat}_n(\mathbb{F}_q) \text{ invertible} \}$ acts on $V = \mathbb{F}_q^n$.
- Reflections on $G_n$: $g \in G_n$ such that $\text{codim } V^g = 1$.
  
  (i) $\text{diag}(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, I_{n-2})$, or conjugates – (unipotent)
  
  (ii) $\begin{bmatrix} \xi & 0 \\ 0 & T_{n-1} \end{bmatrix}$ with $\xi \in \mathbb{F}_q \setminus \{0, 1\}$, or conjugates – (semisimple)

- Fact. $G_n$ is generated by reflections.

  Proof. Gaussian elimination (Linear Algebra)

- Assigning $\ell\ell(g)$ = minimal length of $g \in G_n$ as products of reflections defines a filtered ring structure on $G_n$

  This induces a filtration on the center of the group algebra $\mathcal{Z}_n(q) := \mathcal{Z}(\mathbb{Z}GL_n(q))$
Reflection filtration on $GL_n(q)$

- $G_n := GL_n(q) = \{ g \in \text{Mat}_n(\mathbb{F}_q) \text{ invertible} \}$ acts on $V = \mathbb{F}_q^n$.
- Reflections on $G_n$: $g \in G_n$ such that $\text{codim } V^g = 1$.
  
  (i) $\text{diag} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, l_{n-2}$, or conjugates – (unipotent)
  
  (ii) $\begin{pmatrix} \xi & 0 \\ 0 & T_{n-1} \end{pmatrix}$ with $\xi \in \mathbb{F}_q \setminus \{0, 1\}$, or conjugates – (semisimple)

- Fact. $G_n$ is generated by reflections.
  
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Conjguacy classes of $GL_n(q)$

- $\Phi = \{\text{irreducible monic polynomials in } \mathbb{F}_q[t]\}\setminus\{t\}$
- $g \in G_n$ gives a $\mathbb{F}_q[t]$-module on $V_g = \mathbb{F}_q^n$, $t \cdot v = gv$
- $\mathbb{F}_q[t]$ is PID $\Rightarrow V_g \cong \bigoplus F_q[t]/(f)^m$, for suitable $f \in \Phi$, $m \geq 1$.
- Then $V_g \cong V_{\lambda} = \bigoplus_{f \in \Phi} \mathbb{F}_q[t]/(f)^{\lambda_i(f)}$, for a multi-partition $\lambda = (\lambda(f))_{f \in \Phi}$, with $\lambda(f) = (\lambda_1(f), \lambda_2(f), \ldots)$ such that $n = \|\lambda\| := \sum_f |\lambda(f)|$; $\lambda$ is the type of $g$
- Basic fact. Conjugacy classes of $G_n \iff \{\lambda \mid \|\lambda\| = n\}$
- For $f = t^d - \sum_{i=1}^d a_i t^{i-1} \in \Phi$,

$$J(f) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_d \end{bmatrix}, \quad J_m(f) = \begin{bmatrix} J(f) & I_d & 0 & \cdots & 0 & 0 \\ 0 & J(f) & I_d & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & J(f) & I_d \\ 0 & 0 & 0 & \cdots & 0 & J(f) \end{bmatrix}$$
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0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
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\end{bmatrix}, \quad J_m(f) = \begin{bmatrix}
J(f) & I_d & 0 & \cdots & 0 & 0 \\
0 & J(f) & I_d & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & J(f) & I_d \\
0 & 0 & 0 & \cdots & 0 & J(f)
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Stability for $GL_n(q)$

Conjgucy classes of $GL_n(q)$

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\vdots & \vdots & \vdots & \ddots & \vdots \\
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\end{bmatrix}, \quad J_m(f) = \\
\begin{bmatrix}
J(f) & l_d & 0 & \cdots & 0 \\
0 & J(f) & l_d & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
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Modified type

- Let \( g \in G_n \) be of type \( \lambda \). Define modified type \( \tilde{\lambda} = (\tilde{\lambda}(f))_{f \in \Phi} \):
  
  (i) \( \tilde{\lambda}(f) = \lambda(f) \), for \( f \neq t - 1 \)
  
  (ii) \( \tilde{\lambda}(t - 1) = \"\lambda(t - 1) with 1^{st} column removed\" \) (as for \( S_n \))

- Fact. \( \ell\ell(g) = \|\mu\| \), for \( g \) of modified type \( \mu \)

Example

The lengths of the following matrices are \( d \) and \( d - 1 \), respectively:

\[
J_f = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
a_1 & a_2 & a_3 & \cdots & a_d
\end{bmatrix}, \quad J_m(t - 1) = \begin{bmatrix}
1 & 1 & 0 & \cdots & 0 \\
0 & 1 & 1 & \cdots & 0 \\
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0 & 0 & 0 & \cdots & 1 \\
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Stability for $GL_n(q)$

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0 & 1 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
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Modified type, II

- [Huang-Lewis-Reiner ’17]

(i) \( \ell \ell(g) = \text{codim } V^g \)

(ii) Let \( \lambda, \mu, \nu \) be the modified types of \( g, h, gh \). If \( \|\lambda\| + \|\mu\| = \|\nu\| \), then \( V^g \cap V^h = V^{gh} \) and \( V = V^g + V^h \)
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Stability for $GL_n(q)$

Stable structure constants

- $K_\lambda(n)$: class sum of elements in $GL_n(q)$ of modified type $\lambda$ (if $\|\lambda\| + \ell(\lambda(t - 1)) \leq n$); otherwise $= 0$.

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for $a^\nu_{\lambda\mu}(n) \in \mathbb{N}$.

**Theorem 1 (W-Wang’18)**

1. $a^\nu_{\lambda\mu}(n) = 0$ unless $\|\nu\| \leq \|\lambda\| + \|\mu\|$

2. If $\|\nu\| = \|\lambda\| + \|\mu\|$, then $a^\nu_{\lambda\mu}(n) = a^\nu_{\lambda\mu}$ is independent of $n$.

Proof uses a [remarkable] normal form for triples $(g, h, gh)$ of modified type $\lambda, \mu, \nu$ with $\|\nu\| = \|\lambda\| + \|\mu\|$.

**Remark**

[Méliot’14] $a^\nu_{\lambda\mu}(n)$ is polynomial in $q^n$. (His formulation does not use the modified type or filtration length.)
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Stability of symmetric groups

Stability for $GL_n(q)$

Conjectures and questions

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A stable ring

**Theorem 2 ([W-Wang], a reformulation)**

1. $\mathbb{Z}(\mathbb{Z}GL_n(q))$ is a filtered ring with $\ell\ell(K_\lambda(n)) = \|\lambda\|$.
2. The associated graded $\mathbb{Z}^{gr}(\mathbb{Z}GL_n(q))$ has structure constants independent of $n$:
   \[ K_\lambda(n)K_\mu(n) = \sum_{\|\nu\| = \|\lambda\| + \|\mu\|} a^\nu_{\lambda\mu} K_\nu(n). \]
3. There exists a stable center $G(q) := "\mathbb{Z}^{gr}(\mathbb{Z}GL_\infty(q))"$ with basis $\{K_\lambda \mid \lambda \in \text{Par}(\Phi)\}$ and $K_\lambda K_\mu = \sum_{\|\nu\| = \|\lambda\| + \|\mu\|} a^\nu_{\lambda\mu} K_\nu$.
4. There exists an epi $G(q) \longrightarrow \mathbb{Z}^{gr}(\mathbb{Z}GL_n(q))$, $K_\lambda \mapsto K_\lambda(n)$. 
Stability for $GL_n(q)$

Conjectures and questions

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Stability of symmetric groups

Stability for $GL_n(q)$

Conjectures and questions

Stability for $GL_n(q)$

Examples of stable structure constants $a^\nu_{\lambda\mu}$

Example

1. Computed $a^\nu_{\lambda\mu}$ completely when $\|\lambda\| = \|\mu\| = 1$, e.g.,

$$
a^{(2)}_{t-\xi'}_{(1)_{t-\xi}(1)_{t-\eta}} = q \quad \text{if } \xi' \not\in \{\xi, \eta\};
$$

$$
a^{(1,1)}_{(1)_{t-\xi}(1)_{t-\xi}} = q^2 + q
$$

2. $q = 3, \quad x = y = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad h = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}

\Rightarrow \quad [x] [y] = 3 [h] + ...

Let $x' = \text{diag} (x, 1), \quad y' = \text{diag} (y, 1), \quad h = \text{diag} (h, 1)

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Example, II

Example

Let $\lambda = (1)_{t-\xi_1}$, $\mu = (1)_{t-\xi_2} \cup \cdots \cup (1)_{t-\xi_d}$ with distinct $\xi_i$. Then

$$a_{\lambda \mu}^{\lambda \cup \mu} = (2q - 1)^{d-1}.$$  

- Recall 2 types of reflections: semisimple or unipotent. The structure constants in Examples above ignore such differences.
Example, II

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Conjectures and questions

Conjecture I: a polynomial ring

Computations have suggested general patterns. We shall present several conjectures and open problems.

Conjecture I

The stable center $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{G}(q)$ is a polynomial algebra generated by the single cycle class sums $K_{(r)f}$, for all $r \geq 1$ and $f \in \Phi$.

(Analogous statements hold for $S_n$ and wreath products.)
Conjectures and questions

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Conjectures and questions

Independent of supports

• $\forall \lambda \in \text{Par}(\Phi)$, define its support $\Phi(\lambda) = \{ f \in \Phi \mid \lambda(f) \neq \emptyset \}$

• Let $\{\lambda, \mu, \nu\}, \{\tilde{\lambda}, \tilde{\mu}, \tilde{\nu}\}$ with $\|\nu\| = \|\lambda\| + \|\mu\|$.

Assume $\exists$ a degree-preserving bijection

$\Phi(\lambda) \cup \Phi(\mu) \cup \Phi(\nu) \leftrightarrow \Phi(\tilde{\lambda}) \cup \Phi(\tilde{\mu}) \cup \Phi(\tilde{\nu}), f \mapsto \tilde{f}$, s.t.

$\lambda(f) = \tilde{\lambda}(\tilde{f}), \mu(f) = \tilde{\mu}(\tilde{f}), \nu(f) = \tilde{\nu}(\tilde{f}), \forall f$.

(Say the two triples have same configuration)

Conjecture II (Independence of supports)

The structure constants $a^\nu_{\lambda\mu}$ only depend on the configurations of $\{\lambda, \mu, \nu\}$, i.e., $a^\nu_{\lambda\mu} = a^{\tilde{\nu}}_{\tilde{\lambda}\tilde{\mu}}$.

In particular, the structure constants are insensitive to semisimple/unipotent support. (Supported by all/limited examples.)
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Conjectures and questions

Generic/motivic structure constants

**Question.** How does $a_{\lambda\mu}^{\nu}$ depend on $q$?

- Write $\Phi_q = \Phi$ to indicate its dependence on $q$.
- $\Phi_\mathbb{Z}$: set of monic irreducible polynomials in $\mathbb{Z}[t]$ other than $t$.
- Any polynomial in $\mathbb{Z}[t]$ lies in $\mathbb{F}_q[t]$ by reduction modulo $q$.
  ($\forall f(t) \in \Phi_\mathbb{Z}, f(t) \in \Phi_q$ for $q$ any power of a large enough prime.)

**Conjecture III** (Generic/motivic structure constants)

1. Suppose $\lambda, \mu, \nu \in P(\Phi_\mathbb{Z})$. Then $\exists A_{\lambda\mu}^{\nu}(q) \in \mathbb{Z}[q]$ such that $a_{\lambda\mu}^{\nu} = A_{\lambda\mu}^{\nu}(q)$, $\forall q$ with $\Phi_\mathbb{Z}(\lambda), \Phi_\mathbb{Z}(\mu), \Phi_\mathbb{Z}(\nu) \subset \Phi_q$.

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Conjectures and questions

Generic/motivic structure constants

**Question.** How does \( a_{\lambda\mu}^\nu \) depend on \( q \)?

- Write \( \Phi_q = \Phi \) to indicate its dependence on \( q \).
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Integrality (beyond stable centers)

- [Méliot’14] ∃ polynomials $\tilde{p}^{\nu}_{\lambda \mu}(x)$ with rational coefficients such that $a^{\nu}_{\lambda \mu}(n) = \tilde{p}^{\nu}_{\lambda \mu}(q^n)$

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We have $p^{\nu}_{\lambda \mu}(x) \in \mathbb{Z}[x], \ \forall \lambda, \mu, \nu$.

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References

[W-Wang’18] *Stability of the centers of group algebras of* $GL_n(q)$, arxiv:1805.08796

Thank you!