Relations in doubly laced crystal graphs via discrete Morse theory

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(Kashiwara) crystals

Definition

A *crystal* $\mathcal{B}$ is a directed graph that has vertex set $\mathcal{B}$ and edges labeled with $i \in I$ satisfying the following:

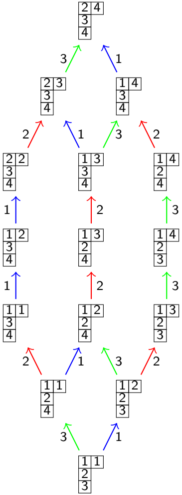
1. all monochromatic directed paths have finite length (no circuits),
2. for every vertex x and color $i \in I$, there is at most one outgoing edge from x labeled i and at most one incoming edge to x labeled i.

If we have an edge $x \xrightarrow{i} y$, we say that $f_i(x) = y$ and $e_i(y) = x$. We call f_i and e_i *crystal operators*. We can define a *covering relation* on a crystal by saying

$$x \lessdot y \iff y = f_i(x)$$

Note: In many interesting cases, this covering relation gives rise to a *partial order*.

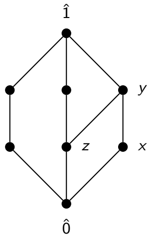
Example:



Type A_3 crystal $\mathcal{B}_{(2,1,1)}$ of shape $\lambda = (2,1,1)$

Poset basics

Example

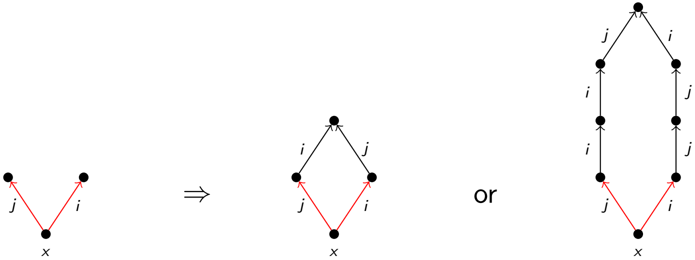


- P has a *minimal element* $\hat{0}$ if $\hat{0} \leq u$ for all $u \in P$.
- P has a *maximal element* $\hat{1}$ if $\hat{1} \geq u$ for all $u \in P$.
- For $u, v \in P$, we say v *covers* u ($u \lessdot v$) if $u < v$ and there is no element $w \in P$ such that $u < w < v$.
- The *open interval* (u, v) is the set $\{x \in P | u < x < v\}$.
- A *saturated chain* from u to v ($u < v$) is a series of cover relations $u = u_0 \lessdot u_1 \lessdot \cdots \lessdot u_k = v$.

Stembridge relations

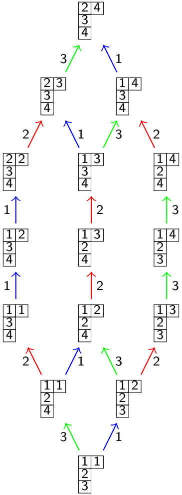
Recall: If $x \xrightarrow{i} y$, we say that $f_i(x) = y$ and $e_i(y) = x$.

Not every crystal arises from a representation, but in the simply laced case Stembridge (2003) gave a list of local structural conditions that characterize when a crystal graph is the crystal of a representation:



The dual picture holds true for the crystal operators e_i and e_j.

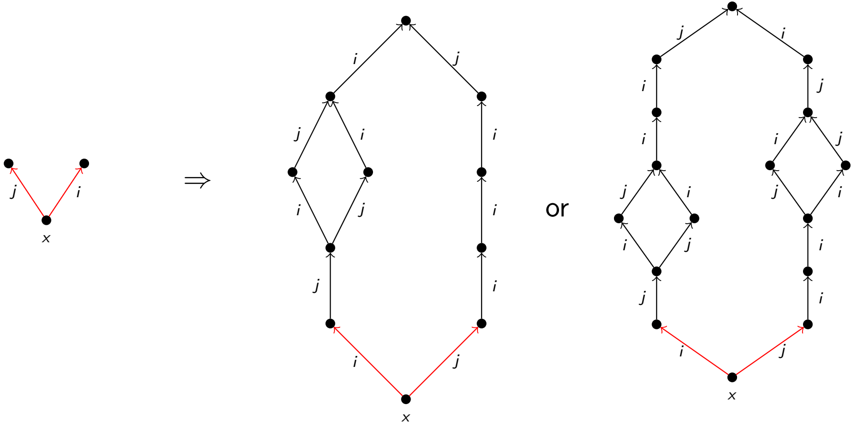
Example:



Type A_3 crystal $\mathcal{B}_{(2,1,1)}$ of shape $\lambda = (2,1,1)$

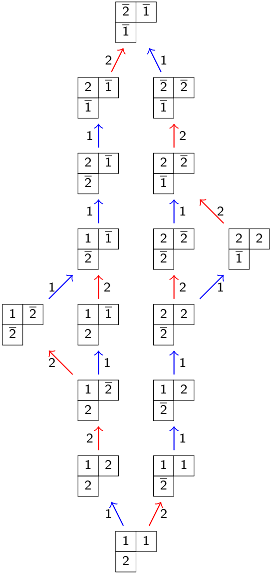
Sternberg relations for the doubly laced case

Sternberg (2006) showed that there are additional relations that hold in doubly laced crystals arising from a representation (B_n, C_n), although these do not characterize doubly laced crystals.



OR previous Stembridge relations.

Sternberg relations



Type C_2 crystal $\mathcal{B}_{(2,1)}$ of shape $\lambda = (2,1)$.

Highest weight representations of type B_2 and C_2

- Recall: A poset P is a *lattice* if for every pair of elements in P there is a unique least upper bound (join) and unique greatest lower bound (meet).
- DKK (2007) showed that crystals of type A_2 are lattices, HL (2017) showed that in general, crystals of type A_n are not lattices.

Theorem (L. (2018))

Crystal posets coming from highest weight representations of type B_2 and C_2 are not lattices.

Proof sketch: The degree five Sternberg relation is asymmetric which will result in incomparable least upper bounds (or incomparable greatest lower bounds).

Möbius function and order complexes

The *Möbius function, μ,* is a combinatorial function that arises as counting coefficients in inclusion-exclusion formulas.

Recursive definition for μ on the interval $[u, v]$:

$$\mu(u,u) = 1,$$
$$\mu(u,v) = -\sum_{u \leq z < v} \mu(u,z)$$

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$$\mu(u,v) = -\sum_{u \leq z < v} \mu(u,z)$$

Definition

The *order complex* of a poset P is the abstract simplicial complex $\Delta(P)$ whose i-dimensional faces are the $(i+1)$-chains $u_0 < u_1 < \cdots < u_i$ in P.

Recall: (Hall, popularized by Rota)

$$\mu(u,v) = \tilde{\chi}(\Delta(u,v))$$

Lexicographic discrete Morse functions

- Discrete Morse theory was introduced by Forman (1998) as a tool to study homotopy type and homology groups of finite CW-complexes.
- Combinatorial reformulation by Chari (2000) where an "acyclic matching" on the face poset of the complex is constructed.
- Babson and Hersh (2005) introduced *lexicographic discrete Morse functions* for the order complex of any finite poset with $\hat{0}$ and $\hat{1}$.
- Use natural edge labeling of crystal graphs to lexicographically order all saturated chains and construct a lexicographic discrete Morse function on the order complex of the poset.

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Theorem (Babson-Hersh (2005))

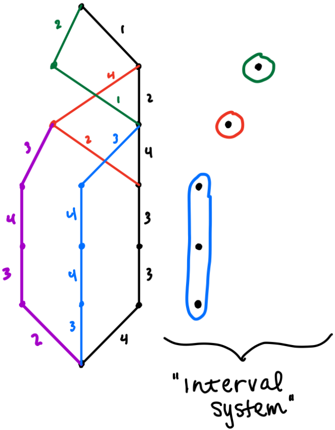
Any edge labeling on any finite poset gives rise to a lexicographic discrete Morse function such that the critical cells give rise to facets whose attachment changes the homotopy type of the complex.

Critical cells and interval system

To see if a facet F_j contributes a critical cell, we look at the *interval system* of F_j.

- Each maximal face in $\overline{F_j} \cap (\cup_{i<j} F_i)$ omits a single interval of $i+1, ..., j-1$, of consecutive ranks. Call the rank interval $[i+1, j-1]$ a *minimal skipped interval* of F_j.
- Call the collection of minimal skipped intervals of F_j the *interval system* of F_j.
- F_j contributes a critical cell if and only if the interval system of F_j covers all ranks in F_j.

Critical cells and interval system



- Looking at interval system for the chain with label sequence $(4, 3, 3, 4, 2, 1)$.
- At each cover relation, check if there is a lexicographically earlier cover relation. If so, travel up to the earliest rank that it reconnects with the black chain. This gives minimal skipped interval.
- We get a critical cell from the lowest element of each interval.
- No critical cell unless interval system fully covers F_j.

A connection between the Möbius function and crystal operators

Theorem (Hersh-Lenart (2017); L. (2018))

Given any $u < v$ in a crystal $\mathcal{B}$ of a highest weight representation of finite simply laced type such that all relations among crystal operators are implied by Stembridge local relations, this implies $\mu(u, v) \in \{-1, 0, 1\}$.

Theorem (L. (2018))

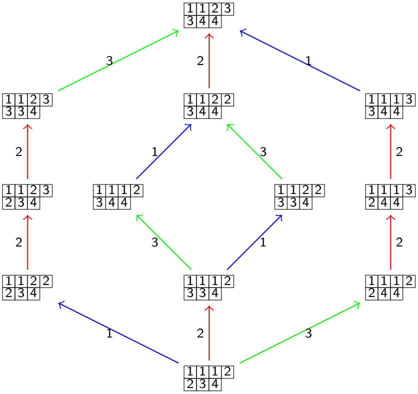
Given any $u < v$ in a crystal $\mathcal{B}$ of a highest weight representation of finite doubly laced type such that all relations among crystal operators are implied by Stembridge and Sternberg local relations, this implies $\mu(u, v) \in \{-1, 0, 1\}$.

Proof sketch: Construct a lexicographic discrete Morse function on the order complex of the interval to see that at most one facet can contribute a critical cell.

Application - simply laced case

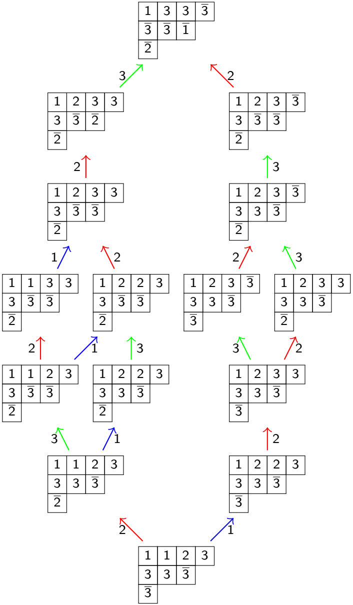
From this result, we know that anytime we find an interval $[u, v]$ in a crystal poset such that $\mu(u, v) \notin \{-1, 0, 1\}$, there must exist relations among crystal operators not implied by Stembridge/Sternberg relations.

- Let $u = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 & 3 & 4 \\ \cline{1-3} \end{array}$, $v = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 3 & 4 & 4 \\ \cline{1-3} \end{array}$
- $\mu(u, v) = 2$.
- The saturated chains with labels $(1, 2, 2, 3)$ and $(3, 2, 2, 1)$ are not connected by Stembridge moves.



Application - doubly laced case

- Let $u = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 3 & 3 & \bar{3} \\ \cline{1-3} \bar{3} \\ \cline{1-1} \end{array}$, $v = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 3 & \bar{3} \\ \hline \bar{3} & \bar{3} & \bar{1} \\ \cline{1-3} \bar{2} \\ \cline{1-1} \end{array}$ in $\mathcal{B}_{(4,3,1)}$ of shape $\lambda = (4, 3, 1)$.
- Contained in an interval where $\mu(u, v) = 2$.
- New relation not implied by Stembridge/Sternberg relations.



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Thank you!

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