Traces of tensor product categories

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Categorification

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E.g.:

$$\text{gVect}_{\mathbb{C}} \to \mathbb{N}[q, q^{-1}]$$

$$V \mapsto \sum_{k \in \mathbb{Z}} q^k \dim(V_k)$$

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Decategorification: Grothendieck group

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Definition

$K_0(\mathcal{C})$ is the abelian group generated by $\{[M] | M \in \mathrm{Ob}(\mathcal{C})/\cong\}$, subject to the relation:

$$\exists \text{ s.e.s. } 0 \to L \to M \to N \to 0$$

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$$K_0(\mathrm{Vect}_{\mathbb{C}}) \cong \mathbb{Z}$$

$$K_0(\mathrm{gVect}_{\mathbb{C}}) \cong \mathbb{Z}[q, q^{-1}]$$

Trace decategorification

The trace (or zeroth Hochschild homology) of a $\mathbb{C}$-linear additive category $\mathcal{C}$:

$$\mathrm{Tr}(\mathcal{C}) := \left(\oplus_{x \in \mathrm{ob}(\mathcal{C})} \mathrm{End}_{\mathcal{C}}(x) \right) \Big/ \mathrm{Span}\{fg - gf\},$$

where f and g run through all pairs of morphisms $f : x \to y$ and $g : y \to x$ with $x, y \in \mathrm{Ob}(\mathcal{C})$.

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$\mathcal{C}$ monoidal $\Rightarrow$ $\mathrm{Span}\{fg - gf\}$ is ideal.
$\Rightarrow$ $\mathrm{Tr}(\mathcal{C})$ as an algebra.

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which is often injective.
Additional advantage: trace is invariant under taking Karoubi envelope.

Example: categorified quantum groups

[Khovanov-Lauda] and [Rouquier] independently constructed categories $\mathbf{U}(\mathfrak{g})$ such that

$$K_0(\mathbf{U}(\mathfrak{g})) \cong \dot{\mathcal{U}}_q(\mathfrak{g})$$

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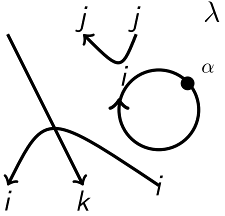
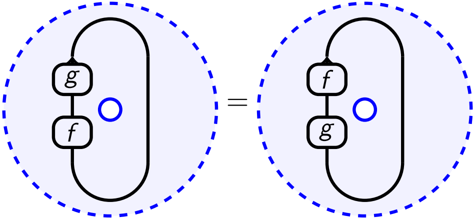
Morphisms given by *KL diagrams*:



modulo relations of the *quiver Hecke algebra*.

Diagrammatic realization of trace

To see trace in diagrams: draw on an annulus.



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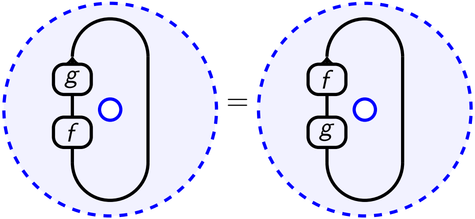
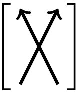


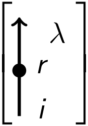
Denote by brackets an element's image in trace, e.g.

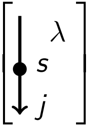


[Beliakov-Habiro-Lauda-Webster]: for $\mathfrak{g}$ simply laced,

$$\mathrm{Tr}(\mathbf{U}(\mathfrak{g})) \cong \dot{\mathcal{U}}(\mathfrak{g}[t]).$$

$\dot{\mathcal{U}}(\mathfrak{g}[t])$ - idempotent form of current algebra.

$(\mathsf{E}_i \otimes t^r)1_\lambda \longmapsto$ 

, $(\mathsf{F}_j \otimes t^s)1_\lambda \longmapsto$ 

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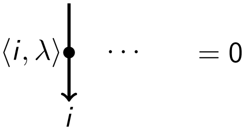
Irreducible $\dot{U}_q(\mathfrak{g})-modules$	$\leftarrow\!\shortmid$	Cyclotomic quotient
$V(\lambda)$	K_0	$\mathbf{U}^{\lambda}$



Categorifying modules

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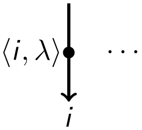

$= 0$

[BHLW] $\mathfrak{g}$ simply laced:

$$\mathrm{Tr}(\mathbf{U}^{\lambda}) = W(\lambda) \text{ (local Weyl module for } \mathcal{U}(\mathfrak{g}[t])\text{).}$$

Deformed cyclotomic quotient $\mapsto \mathbb{W}(\lambda)$ (global Weyl module)

Categorifying tensor products

Let $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n)$ be a sequence of dominant weights.

[Webster] Constructed categories $\mathcal{T}(\underline{\lambda})$ such that

$$K_0(\mathcal{T}(\underline{\lambda})) = V(\underline{\lambda}) = V(\lambda_1) \otimes \ldots \otimes V(\lambda_n)$$

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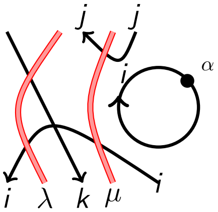
Can be used to prove nondegeneracy of categorified quantum groups for symmetrizable root data.

Stendhal diagrams

Morphisms in $\mathcal{T}$ are given by *Stendhal diagrams.*



Red strands labeled by dominant weights.

Goal

We aim to prove:

Theorem

For $\mathfrak{g}$ simply laced, there is an algebra isomorphism

$$\mathrm{Tr}(\mathcal{T}(\underline{\lambda})) \longrightarrow W(\underline{\lambda}) = W(\lambda_1) \otimes \ldots \otimes W(\lambda_n)$$

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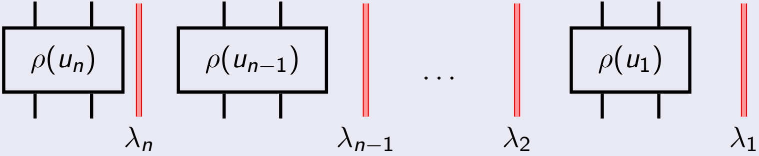
The trace of a deformed version is isomorphic to $\mathbb{W}(\underline{\lambda})$.

Constructing the map

Lemma

The map $W(\underline{\lambda}) \to \mathrm{Tr}(\mathcal{T}(\underline{\lambda}))$

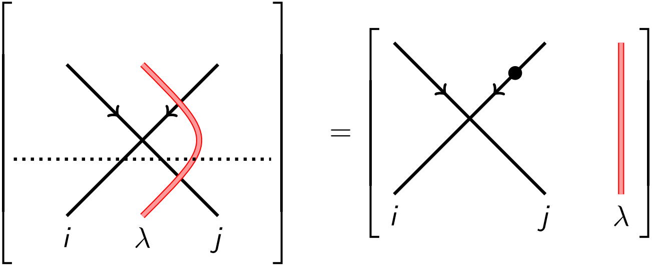
$$u_n(\cdots u_2((u_1 w_{\lambda_1}) \otimes w_{\lambda_2}) \otimes \cdots \otimes w_{\lambda_n}) \longmapsto$$



is an algebra homomorphism (ρ is the isomorphism from BHLW).

Surjectivity

We show that $\mathrm{Tr}(\mathcal{T}(\underline{\lambda}))$ is spanned by Stendhal diagrams with no red-black crossings:



These are clearly in the image of the map.

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Deform category so that special point is the trace, and generic point has a known dimension.

Selected references

- [Beliakov-Habiro-Lauda-Webster] *Current algebras and categorified quantum groups.* 2014.
- [Khovanov-Lauda] *A diagrammatic approach to categorification of quantum groups I-III*. 2008.
- [Rouquier] *Quiver Hecke algebras and 2-Lie algebras*. 2011
- [Webster] *Knot invariants and higher representation theory*. 2013
- [Webster] *Unfurling Khovanov-Lauda-Rouquier algebras.* 2016