Quiver Varieties and Symmetric Pairs

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REPRESENTATION THEORY
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Motivations: Schur dualities
—based on joint works with H. Bao, J. Kujawa and W. Wang.
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- Results on nilpotent Slodowy slices
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Overview:

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  —based on joint works with H. Bao, J. Kujawa and W. Wang.
- Results on nilpotent Slodowy slices
- The construction of iQuiver Varieties (iQV)
- Connection with real classical groups
Schur duality and its generalizations.

- Schur duality: $\mathcal{U}(\mathfrak{gl}_n) \curvearrowright \mathbb{C}^n \otimes d \curvearrowleft \mathbb{C}[S_d]$. (Schur ~1901)

(Schur duality) $\mathcal{U}_q(\mathfrak{gl}_n) \curvearrowright \mathbb{T}_n, d \curvearrowleft \mathbb{H}_d$, with $\mathbb{T}_n, d = \mathbb{C}(q^n) \otimes d$.

(Jimbo, 1986)

Schur duality: $\mathcal{U}_\sigma(q)(\mathfrak{gl}_n) \curvearrowright \mathbb{T}_n, d \curvearrowleft \mathbb{H}_{\mathbb{B}/\mathbb{C}/\mathbb{D}_d}$. (Green 1997, Bao-Wang 2013)
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- iSchur duality:  \( U^\sigma_q(\mathfrak{gl}_n) \bowtie T_{n,d} \bowtie H_d^{B/C/D} \). (Green 1997, Bao-Wang 2013)
Compatibility and quantum symmetric pairs

\[ U_q^\sigma(\mathfrak{gl}_n) \rightarrow T_{n,d} \rightarrow H^{B/C/D} \rightarrow \text{iSchur} \]

\[ U_q(\mathfrak{gl}_n) \rightarrow T_{n,d} \rightarrow H_d \rightarrow \text{Schur} \]

The pair \((U_q(\mathfrak{gl}_n), U_q^\sigma(\mathfrak{gl}_n))\) is a quantum symmetric pair.

These algebras were studied previously by Noumi, Letzter, Kolb, etc.
Compatibility and quantum symmetric pairs

$U_q(\mathfrak{gl}_n) \rightarrow T_{n,d} \rightarrow H^{B/C/D} \rightarrow i\text{Schur}$

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A symmetric pair \((g, g^\theta)\) is a complex semisimple Lie algebra \(g\) together with its fixed-point subalgebra \(g^\theta\) under an involution \(\theta\).
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When specialized to \(q = 1\), the pair \((U_q(g\mathfrak{t}_n), U_q^\sigma(g\mathfrak{t}_n))\) is of type \(A_{iii}/A_{iv}\), with Satake diagram (all vertices are white):

\[
\theta : \quad \circ \quad \circ \quad \circ \quad \ldots \quad \circ \quad \circ \quad \circ
\]
Symmetric pairs

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Classification of symmetric pairs goes back to É Cartan, and is equivalent to the classification of real simple Lie algebras, via Satake diagrams= bicolor Dynkin diagrams with diagram involution.

When specialized to \(q = 1\), the pair \((U_q(gl_n), U^\sigma_q(gl_n))\) is of type \(A_{iii}/A_{iv}\), with Satake diagram (all vertices are white):

\[
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\]

Caution: \(U^\sigma_q(gl_n)\) NOT a fixed-point subalgebra of \(U_q(gl_n)\).
iSchur duality is used by Bao-Wang to solve Kazhdan-Lusztig problem for $\mathfrak{osp}_{m|n}$, following Brundan’s approach to $\mathfrak{gl}_{m|n}$. 

Howe duality is used by Ehrig-Stroppel to solve problems for $\mathfrak{so}_{2m}$. 

Bao-Wang’s work contains an icanonical basis (iCB for short) for representations of coideal subalgebras of quantum $\mathfrak{sl}_n$.

Bao-Wang’s work on iCB has been extended by themselves to any coideal subalgebras of quantum groups of finite type.
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Applications/Why do we care?

- iSchur duality is used by Bao-Wang to solve Kazhdan-Lusztig problem for $\mathfrak{osp}_{m|n}$, following Brundan’s approach to $\mathfrak{gl}_{m|n}$.
- Howe duality is used by Ehrig-Stroppel to solve problems for $\mathfrak{so}_{2m}$.
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Geometric Schur duality

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- BKLW obtained an iCB for quantum symmetric pairs of type \( \text{A}_{iii/iv} \). A slight modification leads to remarkable positivity (L.-Wang, Fan-L.). (weak categorification)
- Genuine categorification of (part of ) BKLW’s work has been done by H. Bao, P. Shan, W. Wang and B. Webster.
There is a ‘type $A$’ line of research:

\[ \mathcal{F}_{n,d} \hookrightarrow T^* \mathcal{F}_{n,d} \hookrightarrow \text{Nakajima varieties} \]

$\mathfrak{sl}_n$ dual $\mathfrak{g}_{ADE}$
There is a ‘type A’ line of research:

\[
\mathcal{F}_{n,d} \rightsquigarrow T^*\mathcal{F}_{n,d} \rightsquigarrow \text{Nakajima varieties}
\]

\[
\mathfrak{sl}_n \quad \text{dual} \quad \mathfrak{g}_{\text{ADE}}
\]

In light of the previous work, we should have a line of research, based on “classical type” geometry:

\[
\mathcal{F}^\sigma_{n,d} \rightsquigarrow T^*\mathcal{F}^\sigma_{n,d} \rightsquigarrow \text{iQV???
}
\]

\[
\mathfrak{sl}_n^\sigma \quad \text{dual} \quad \mathfrak{g}^\theta_{\text{ADE}}
\]

The existence of iQV is also conjectured by Weiqiang Wang.
The simple answer to construct iQV:

**Analogue**

\[ M_\zeta(w) \sim \theta \Rightarrow g \]

Answer: \( iQV = M_\zeta(w) \sigma : \) the fixed point locus of \( M_\zeta(w) \) under an \( \sigma \).
A simple answer

The simple answer to construct iQV:

An Analogue

\[ M_\zeta(w) \rightsquigarrow g \]
\[ iQV?? \rightsquigarrow g^\theta, \text{ for } \theta \in \text{Aut}(g) \]
The simple answer to construct $iQV$:

**Analogue**

\[
\mathcal{M}_\zeta(\mathbf{w}) \rightsquigarrow g \\
iQV \rightsquigarrow g^\theta, \text{ for } \theta \in \text{Aut}(g)
\]

Answer: $iQV = \mathcal{M}_\zeta(\mathbf{w})^\sigma$: the fixed point locus of $\mathcal{M}_\zeta(\mathbf{w})$ under an $\sigma$. 
We have the following comparison of results in QV and iQV.

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**Caution:** the automorphism $\sigma$ is not always an involution. For the Weyl group action of type $G_2$, $\sigma$ is of order 6.
One of the rectangular symmetries reads as follows

\[ \widetilde{S}^{sp}_{\mu', \lambda} \xrightarrow{\sigma \pi} \widetilde{S}^{o}_{\nu', \lambda'} \]

\[ S^{sp}_{\mu', \lambda} \xrightarrow{\sigma \pi} S^{o}_{\nu', \lambda'} \]

where each pair \((\mu', \tilde{\mu}')\) and \((\lambda, \tilde{\lambda})\) can be fit into a rectangle:

\[ \ell(\lambda) \]

\[ n + 1 \]
As a special case of the rectangular symmetry, we recover

Henderson-Licata, 2013

\[ S^{sp}_{n, k (n-k)} \cong S^{0}_{1, (n+1), (k+1) (n+1-k)} . \]
Special case: two-row Slodowy slices

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Henderson-Licata, 2013

\[ S_{n^1,k^1(n-k)}^{\text{sp}} \cong S_{1^1(n+1)^1,(k+1)^1(n+1-k)}^{0n+2}. \]

Simultaneously, we also deduce

Wilbert, 2015; Ehrig-Stroppel, 2013

\[ B_{e_0}^{\text{sp}} \cong B_{e_0}^{50n+2} \quad \text{— Springer fibers of the associated Slodowy slices.} \]
Special case: two-row Slodowy slices

As a special case of the rectangular symmetry, we recover

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\[ B_{\tilde{e}_0}^{\text{sp}} \cong B_{\tilde{e}_0}^{\text{50}n+2} \quad \text{— Springer fibers of the associated Slodowy slices.} \]

And we solve a conjecture of Henderson-Licata for free:

\[ \tilde{S}_{n^1,k^1(n-k)^1}^{\text{sp}} \cong \tilde{S}_{1^1(n+1)^1,(k+1)^1(n+1-k)^1}^{0n+2}. \]
We have $S^{sp}_{\tilde{\mu}', \tilde{\lambda}} \cong S^o_{\mu', \lambda}$, if the partitions are related as follows.
We have $S^{sp|\bar{\lambda}|}_{\bar{\mu}',\bar{\lambda}} \cong S^{sp|\lambda|}_{\mu',\lambda}$, if

\begin{align*}
\mu' & \oplus \lambda = \bar{\mu}' \\
\lambda & \oplus \mu' = \bar{\lambda}
\end{align*}
One starts with a quiver $Q$ with underlying graph $\Gamma$, and its framed version $Q^f$ by adding an extra copy of vertex set and arrow connecting to the original vertices. For example

\[ Q = \bullet \rightarrow \bullet, \quad Q^f = \downarrow \quad \downarrow \]

\[ \bullet \quad \bullet \]

\[ \bullet \quad \rightarrow \quad \bullet \]
Consider the geometries:

\[ \text{rep } Q \xrightarrow{T^*_{\text{rep } Q}} \text{Hamilt. reduction } T^*_{\text{rep } Q^f} / G. \]

Each step yields rich geometries and contains much representation theoretical information for the Lie algebra \( \Gamma \) associated to \( \Gamma \).
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\[ \text{rep} Q^f \]

Each step yields rich geometries and contains much representation theoretical information for the Lie algebra \( g_\Gamma \) associated to \( \Gamma \).
Specifically, consider the cotangent space $T^* \text{rep} Q_{v,w}^f$ of representations of $Q_{v,w}^f$ of fixed dimension vectors $v, w$. There is a (reductive/gauge) group $G_v$ acts nicely on $T^* \text{rep} Q_{v,w}^f$. General machinery in symplectic geometry says that there is a moment map

$$\mu : T^* \text{rep} Q_{v,w}^f \rightarrow (\text{Lie } G_v)^*.$$
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Nakajima (quiver) variety is defined to be the Hamiltonian reduction

$$\mathcal{M}_{\zeta}(v, w) = \mu^{-1}(\zeta_C)\,\sslash\,\zeta_R G_v, \quad \zeta = (\zeta_C, \zeta_R).$$
Rank one: $\zeta = (0, 1)$ or $(0, 0)$

In rank one case,

$$T^*_{\text{rep}} Q_{v,w}^f = \text{Hom}(\mathbb{C}^w, \mathbb{C}^v) \oplus \text{Hom}(\mathbb{C}^v, \mathbb{C}^w)$$

and the fiber at 0 of the moment map is given by

$$\mu^{-1}(0) = \{(p, q) | pq = 0\}.$$
In rank one case,

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**Rank one Nakajima variety**

Nakajima varieties are

\[ M_{(0,1)}(v, w) = \{(p, q) \in \mu^{-1}(0) | q \text{ injective}\}/GL(\mathbb{C}^v), \text{(GIT quotient)} \]

\[ M_{(0,0)}(v, w) = \mu^{-1}(0)//GL(\mathbb{C}^v) \text{ (categorical quotient)} \]
Nakajima varieties and cotangent bundle of Grassmannian

The assignment \((p, q) \mapsto (qp, \text{im}(q))\) identifies Nakajima varieties with the cotangent bundle of Grassmannian and its affinization.

\[
\begin{align*}
\mathcal{M}_{(0,1)}(v, w) & \cong T^* \text{Gr}(v, w) \\
\downarrow & \downarrow \text{Springer resolution} \\
\mathcal{M}_{(0,0)}(v, w) & \rightarrow \{ x \in \text{End}(\mathbb{C}^w) | x^2 = 0, \ldots \}
\end{align*}
\]

Ginzburg’s setting

In general, the cotangent bundle \(T^* \mathcal{F}_{n,d}\) used in Ginzburg’s construction is a very special case of Nakajima varieties of type \(A\).
Isomorphisms on quiver varieties

Let \( a : \Gamma \to \Gamma \) be a diagram automorphism.

**Naive diagram isomorphism**

\[
a : \mathcal{M}_\zeta(v, w) \to \mathcal{M}_{a(\zeta)}(a(v), a(w)).
\]
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$a = 1.$
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Associate to each $C^v_i$ and $C^w_i$ a non-degenerate bilinear form.
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Associate to each \( C^v_i \) and \( C^w_i \) a non-degenerate bilinear form. One can define automorphisms, via taking adjoints, on Nakajima’s varieties:

Isomorphism \( \tau_{\zeta} \)

\[ \tau_{\zeta} : M_{\zeta}(v, w) \to M_{-\zeta}(v, w). \]
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\tau_\zeta : M_\zeta(v, w) \to M_{-\zeta}(v, w).
\]

**Rank one**

\[
\tau_\zeta : (p, q) \mapsto (-q^*, p^*) \pmod{G_v}.
\]
Recall $\mathcal{W}_\Gamma$ be the Weyl group of $\Gamma$.

**Reflection functors $S_\omega$ of Nakajima, Lusztig and Maffei**

$S_\omega : M_\zeta(v, w) \rightarrow M_{\omega(\zeta)}(\omega * w v, w), \forall \omega \in \mathcal{W}_\Gamma$. 
Recall $W_{\Gamma}$ be the Weyl group of $\Gamma$.

**Reflection functors $S_\omega$ of Nakajima, Lusztig and Maffei**

$S_\omega : \mathcal{M}_\zeta(v, w) \to \mathcal{M}_{\omega(\zeta)}(\omega \ast_w v, w), \forall \omega \in W_{\Gamma}$.

**Rank one**

$S_i : (p, q) \mapsto (p', q'), \mathbb{C}^v \xrightarrow{q} \mathbb{C}^w \xrightarrow{p'} \mathbb{C}^{w-v}$ is exact and $qp = q'p'$. 
Taking the composition of the above three isomorphisms yields:

**Isomorphism \( \sigma \)**

\[
\sigma \equiv \sigma_{a,\zeta,\omega} := aS_{\omega} \tau_{\zeta} : \mathcal{M}_{\zeta}(v, w) \to \mathcal{M}_{-a\omega(\zeta)}(a(\omega *_{w} v), aw).
\]
Taking the composition of the above three isomorphisms yields:

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**Definition of iQV**

\[
\mathcal{M}_{\zeta}(v, w)^{\sigma}, \quad \text{if } \zeta = -a\omega(\zeta), \quad v = a(\omega \ast w v) \text{ and } w = aw.
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It comes equipped with a projective morphism:

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**Quiver variety is an iQV**

\[
M^\Gamma \times \Gamma(w)^\sigma \cong M_\zeta(w) \quad \text{for } \sigma = a\tau_\zeta.
\]
The assignment \((p, q) \mapsto (qp, \text{im}(q))\) identifies iQV with the cotangent bundles of maximal isotropic Grassmannians.

\[
\begin{align*}
\mathcal{M}_{(0,1)}(v, w)^\sigma \ & \xrightarrow{\cong} \ T^*\text{Gr}(v, w)^{\sigma'} \\
\downarrow \ & \downarrow \\
\mathcal{M}_{(0,0)}(v, w)^\sigma \ & \xrightarrow{} \ \{x \in \text{End}(\mathbb{C}^w) | x^2 = 0, \ldots\}^{\sigma'}
\end{align*}
\]

Here \(\sigma'\) depends on the form on \(\mathbb{C}^w\).
Proof

The action of $\sigma$ on $(p, q)$ is $(p, q) \xrightarrow{\tau\xi} (-q^*, p^*) \xrightarrow{S_i} (-q^*)', (p^*)')$.
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The action of $\sigma$ on $(p, q)$ is $$(p, q) \xrightarrow{\tau_{\xi}} (-q^*, p^*) \xrightarrow{S_i} (-q'^*, (p^*)').$$

So it sends

$$qp \mapsto -(p^*)'(q^*)' = -p^* q^* = -(qp)^* \leftrightarrow x \mapsto -x^*$$
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\[
qp \mapsto -(p^*)'(q^*)' = -p^* q^* = -(qp)^* \quad \text{and} \quad x \mapsto -x^*
\]

\[
\im q \mapsto \im(p^*)' = \ker q^* \quad \text{and} \quad F \mapsto F^\perp.
\]

$\perp$ is taken with respect to the form on $\mathbb{C}^w$. 
Kraft-Procesi considered an $A_n$ quiver with alternating forms, say:

$$Sp_{w_1}$$

$$O_{v_1} \xrightarrow{} Sp_{v_2} \xrightarrow{} O_{v_3} \xrightarrow{} \cdots$$
Kraft-Procesi considered an $A_n$ quiver with alternating forms, say:

$$\text{Sp}_{w_1} \xrightarrow{\mu^{-1}(0)^\sigma} G_v^\sigma = O_{v_1} \times \text{Sp}_{v_2} \times O_{v_3} \times \cdots$$

Consider the fixed-points $\mu^{-1}(0)^\sigma$, $G_v^\sigma = O_{v_1} \times \text{Sp}_{v_2} \times O_{v_3} \times \cdots$. 
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\[
\begin{align*}
O_v^1 \quad & \quad \quad & \quad Sp_v^2 \quad & \quad \quad & \quad O_v^3 \quad & \quad \quad & \quad \cdots
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\]

Consider the fixed-points $\mu^{-1}(0)\sigma$, $G_v^\sigma = O_v^1 \times Sp_v^2 \times O_v^3 \times \cdots$.

Theorem (Kraft-Procesi, 1982): classical nilpotent orbits

\[
\mu^{-1}(0)\sigma \, / \, / \, G_v^\sigma \cong \overline{O_{\mu'}}^{spw_1}, \text{ for certain } v, w \text{ and the associated } \mu'.
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**Theorem (Kraft-Procesi, 1982):** classical nilpotent orbits

$\mu^{-1}(0)^\sigma / / G^\sigma_v \cong \overline{O}_{\mu'}^{sp w_1}$, for certain $v$, $w$ and the associated $\mu'$.

**Theorem: iQV and Kraft-Procesi**

In Kraft-Procesi’s setting, $\mathcal{M}_0(v, w)^\sigma \cong \mu^{-1}(0)^\sigma / / G^\sigma_v$, for $a = 1$.
Geometric invariant theory does not seem to apply to Kraft-Procesi’s approach: essentially no non-trivial character of $G_v^\sigma$.

Theorem: Analogue of Ginzburg: Cotangent bundles of isotropic flag varieties in Kraft-Procesi’s setting, we get $M_\zeta(v,w) \cong T^*F_{sp}^1(v,w)$, if $\zeta = (1,0)$, $\omega = \omega_0$.

where $\omega_0$ is the longest element in $W_G$ and $a = 1$. 
GIT in Kraft-Procesi’s approach (one of my mental blocks)

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Nakajima’s generalization: nilpotent Slodowy slices

In more general type-$A$ setting of Kraft-Procesi, say:

\[
\begin{align*}
\text{Sp}_{w_1} & \quad \text{O}_{w_2} & \quad \text{Sp}_{w_3} \\
\text{O}_{v_1} & \quad \text{Sp}_{v_2} & \quad \text{O}_{v_3}
\end{align*}
\]

\[\sigma \mapsto S_{\text{sp}_{\mu'}, \lambda}, \text{or} \quad S_{\text{so}_{\mu'}, \lambda},\]

where $S_{\text{sp}_{\mu'}, \lambda}$ is a nilpotent Slodowy slice in $\text{sp}_{\tilde{w}_1}$.

Proposition

In the above setting, there is a closed immersion (isomorphism expected):

\[\mu - 1(0)_{\sigma} \mapsto S_{\text{sp}_{\mu'}, \lambda},\]

which relies on a result of quiver-analogue of classical invariants.
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Nakajima asserted at several places that

\[
\mu^{-1}(0)^{\sigma} / G^\sigma \hookrightarrow \text{Sp}^{\mu',\lambda} \text{ or } \text{Sp}^{\sigma,\mu',\lambda},
\]

where \( \text{Sp}^{\mu',\lambda} = \overline{O}^{\mu'} \cap S_{\lambda} \cap \text{Sp}_{\tilde{w}^1} \) is a nilpotent Slodowy slice in \( \text{Sp}_{\tilde{w}^1} \).
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\end{array}\]

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\[\mu^{-1}(0)^\sigma // G^\sigma \hookrightarrow \text{Sp}_{\mu',\lambda}^o \text{ or } \text{Sp}_{\mu',\lambda}^o\]

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Theorem: Partial Resolutions of nilpotent Slodowy slices

Moreover, in the above type A Dynkin diagram setting, we have

\[
\mathcal{M}_\zeta(\mathbf{v}, \mathbf{w})^\sigma \xrightarrow{\pi^\sigma} \tilde{S}^{sp}_{\mu', \lambda} \xleftarrow{T^* \mathcal{F}^{sp}_{\tilde{v}, \tilde{w}}} \mathcal{S}^{sp}_{\mu', \lambda} \xrightarrow{\Pi} \text{sp}_{\tilde{w}}
\]

(\text{where } \tilde{\mathbf{w}} = \sum_{i \in I} i \mathbf{w}_i, \tilde{\mathbf{v}} = \mathbf{v}_i + \sum_{j \geq i} (j - i) \mathbf{w}_j.\)

This result leads to previous applications in classical geometries.
Instantons on ALE space and Nakajima varieties of type A, due to Nakajima

**Unitary**

Regular part of Nakajima varieties = unitary instantons on ALE spaces.

It is known to Nakajima that

**Classical type**

Regular part of iQV (of some $\sigma$) = $SP/\text{SO}$ instantons on ALE spaces. (arxiv: 1801.06286.)
Now we return to the general setting:

Proposition: Independence of forms on $V$

The $\mathcal{M}_\zeta(v, w)^\sigma$ is independent of choices of forms on $V$.
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From now on, forms on $W$ are either *orthogonal* or *symplectic*.

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Proposition: Weyl group action

Let $\mathcal{W}^{a\omega}_\Gamma = \{x \in \mathcal{W}_\Gamma | x\omega = \omega x, a(x) = x \}$. There exists a $\mathcal{W}^{a\omega}_\Gamma$-action:

$$\mathcal{W}^{a\omega}_\Gamma \curvearrowright H^*(\mathcal{M}_\zeta(v, w)^\sigma), \quad w - C_\Gamma v = 0.$$ 

$\mathcal{W}^{a\omega}_\Gamma$ includes Weyl groups of $B_\ell/C_\ell/F_4/G_2$ types.
Conjectures

Conjecture

There is an action

\[ g^\theta \curvearrowright H^*(\mathcal{M}_\zeta(w)^\sigma), \quad (\zeta \text{ generic}) \]

where \((g, g^\theta)\) for a symmetric pair of type \(Ai, Aiii, Di, Ei, Eii, Ev, Eviii\), whose Satake diagram has no black dots. Note \(g^\theta\) of type \(Ai\) is \(so_n\).

It holds for \(Aiii/Aiv\). There are several supporting evidence.
Symmetric pairs have been pervasive in the study of representations of real simple/reductive groups.

**QV and real simple groups**

Does QV/iQV have more direct connections with real classical groups?
To any symmetric pair \((g, g^\theta)\), it yields a complex Cartan decomposition

\[ g = g^\theta \oplus p, \]

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Quiver model of \(\mathcal{N}(p)\): Lagrangian version of iQV

The anti-symplectic version, say \(\mathcal{M}_\zeta(w)^\sigma\), of iQV yields a quiver/linear model of \(\mathcal{N}(p)\) and associated Slodowy slices. Almost all results from symplectic version have an anti-symplectic counterpart, such as rectangular symmetry etc. (But not the semismallness of projection from \(\pi\).)
This correspondence works for all symmetric pairs \((\mathfrak{g}, \mathfrak{g}^\theta)\) of classical type.
A correspondence

\[ g \quad \uparrow \quad g^\theta \quad \uparrow \quad p \]

Nakajima varieties \quad Symplectic subvarieties \quad Lagrangian subvarieties

This correspondence works for all symmetric pairs \((g, g^\theta)\) of classical type. Note that \(N(p) \cong N(G_\mathbb{R})\), the Kostant-Sekiguchi homeomorphism of Chen-Nadler. It is reasonable to expect the same holds in quiver setting.

**Kostant-Sekiguchi correspondence for quiver varieties**

There should be a homeomorphism \(\mathcal{M}_0(w)_\mathbb{R} \cong \mathcal{M}_0(w)^{\hat{}}\).
Representations of symmetric pairs

Through the work of Harish-Chandra, the following two classes of representations are the same (very roughly speaking).

- Unitary representations of $G_\mathbb{R}$, a real simple group.
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A shadow of the $(g, K)$-module: Harish-Chandra $(g, \text{Lie}K)$-modules. $\text{Lie}K = g^\theta$ for some $\theta$, studied by Diximier, Lepowsky, Zuckerman, etc.

On the other hand, there is a Yangian $\mathcal{Y} \equiv \mathcal{Y}(g_{\Gamma})$ action on the torus equivariant cohomology $H^*_T(\mathcal{M}_\zeta(w))$ of Nakajima variety, due to Varagnolo (via correspondence), Maulik-Okounkov (via R-matrix).
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**$(g, \text{Lie} K)$-structure in Nakajima varieties**

There is an action of an (affine) symmetric pair $(\mathcal{Y}, \mathcal{Y}_{\sigma}) \curvearrowright H^*_T(M_\zeta(w))$, where $\mathcal{Y}_{\sigma}$ is a twisted Yangian, i.e., an affinization of $U(g_{\Gamma}^\theta)$. 
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How to lift $(g, \text{Lie}K)$-structure to a $(g, K)$-structure remains to be done.
The above result is obtained via Maulik-Okounkov’s R-matrix approach.
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- We show that certain fixed-point subvarieties of $iQV$ is a product of $QV$ of smaller ranks.
- Stable envelopes exist.
- The $R$-matrix (or rather $K$-matrix) satisfies the reflection equation $RKRK = KRKR$. 
The above result is obtained via Maulik-Okounkov’s R-matrix approach.

We show that certain fixed-point subvarieties of iQV is a product of QV of smaller ranks.

Stable envelopes exist.

The $R$-matrix (or rather $K$-matrix) satisfies the reflection equation $RKRK = KRKR$.

The twisted Yangian $\mathfrak{y}_\sigma$ is then constructed using $K$-matrix via Faddeev, Reshetikhin, and Takhtajan’s construction.
Comparison, II

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- $g^\theta \curvearrowright H^\ast(\mathcal{M}_\zeta(\mathbf{w})^\sigma)$ (conj.)
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**LI (SUNY at BUFFALO)**

**Quiver varieties and symmetric pairs**

**October UVA**

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<td>symp. resolution</td>
<td>symp. partial resolution/ lgrngn</td>
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<tr>
<td>$\pi$ semismall</td>
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<td>Weyl groups action of type $ADE$</td>
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Thank you very much!