Recent Progress on the Saturation Conjecture for type D

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Overview

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Fact

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For which triples λ, μ, ν (call them $\mathcal{R}(G)$) is

$$V(\lambda) \otimes V(\mu) \otimes V(\nu)^G \neq (0)?$$

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Analogously,

$$H^0((G/B)^3, L_\lambda \boxtimes L_\mu \boxtimes L_\nu) \simeq [V(\lambda) \otimes V(\mu) \otimes V(\nu)]^\vee.$$

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$$(\lambda, \mu, \nu) + (\lambda', \mu', \nu') = (\lambda + \lambda', \mu + \mu', \nu + \nu')$$

with identity $(0, 0, 0)$.

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and products of nonzero G-invariant sections are nonzero and G-invariant.

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But in general the question is still hard to answer.

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Call this set $\mathcal{C}(G)$; it is also a monoid.

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This yields a description of $\mathcal{C}(G)$ using inequalities.

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$$\lambda(ux_P) + \mu(vx_P) + \nu(wx_P) \leq 0.$$

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Conjecture (Kapovich-Millson)

If G is of simply-laced type (A, D, E), then $\mathcal{R}(G) = \mathcal{C}(G)$.

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Thus we may find desired products by testing

$$P_{w_0 u} \cdot P_{w_0 v} \cdot P_{w_0 w} = P_{w_0} \mod J.$$

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Polynomial manipulation replaced by sums/products of rationals

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$$\vec{\lambda}(j, v) = \left(\sum c_k^{(1)} \omega_k, \sum c_k^{(2)} \omega_k, \sum c_k^{(3)} \omega_k \right),$$

where $c_k^{(i)}$ are certain intersection-theoretic counts.

Slight modification

Can we produce rays directly?

Theorem (Belkale & K.)

Fix a maximal parabolic P and $w_1, w_2, w_3 \in W^P$ so that

$$[X_{w_1}] \odot_0 [X_{w_2}] \odot_0 [X_{w_3}] = [X_e];$$

these define a facet $\mathcal{F}$ of $\mathcal{C}(G)$. For every $v \xrightarrow{\alpha} w_j$ with α simple, there exists an extremal ray $\vec{\lambda}(j, v)$ on $\mathcal{F}$. Remaining rays can be induced from the smaller cone $\mathcal{C}(L^{ss})$ according to an explicit formula.

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Rays formulas were used in a crucial way!

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Ressayre, Richmond, Pasquier, others...version of saturation...

Thank you!